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Thermal scalar quantum field in static background gauge potentials

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Abstract. We study, at finite temperature, the energy–momentum tensor $T_{\mu\nu}(x)$ of a charged scalar field interacting with a static background electromagnetic field. $T_{\mu\nu}$ separates into an UV divergent part $T^{sea}_{\mu\nu}$ representing the virtual sea, and an UV finite part $T^{plasma}_{\mu\nu}$ describing the thermal plasma of the matter field. In the presence of a constant electric field $\vec{E}$, $T^{sea}_{\mu\nu}$ remains uniform while $T^{plasma}_{\mu\nu}$ becomes spatially non-uniform in the direction of $\vec{E}$. A related (periodic) spatial non-uniformity is found for a constant static magnetic field $\vec{B}$ if one spatial direction perpendicular to $\vec{B}$ is compactified to a circle.

1. Introduction

A finite-temperature ($T > 0$) or thermal quantum field can be visualized as a *sea* of virtual particles coexisting throughout space with a thermal *gas* of real particles or field excitations. The virtual particle sea is independent of the temperature T. It can, however, be deformed by coupling the field to static background structures of various kinds. This is generally known as the static vacuum Casimir effect. Somewhat less widely known is that, for boundaries and other static backgrounds, the thermal gas is 'mechanically' distorted along with the sea.

Abelian (and non-Abelian) gauge theories present Casimir problems with very interesting and unusual features arising from the underlying gauge invariance. Indeed, the restrictions on the class of allowed gauge transformations imposed by $T > 0$ are found to have remarkable consequences for the spatial energy distribution of a charged thermal matter field coupled to a static background electromagnetic field ($\vec{E}$, $\vec{B}$). The local distortion by ($\vec{E}$, $\vec{B}$) of the virtual sea and thermal *plasma* (as we now refer to the thermal gas consisting of both particles and antiparticles) can be revealed by computing the stress energy–momentum tensor $T^{(\beta)}_{\mu\nu}$ of the thermal field and the local Lagrangian $\mathcal{L}_\beta$.

The problem of charged quantum fields coupled to a uniform electromagnetic background field is an old one, going back to renowned papers by Euler and Heisenberg [1] and Schwinger [2]. Much of the subsequent literature is reviewed in [3–5], nearly all of this work being restricted to zero-temperature fields. Much less work has been done on $T > 0$ quantum fields coupled to a static electromagnetic background (see, in particular, papers [6–10] and reviews [11, 12]). For some reason the global features of the problem have received most of the attention, while local aspects seem to have been neglected.

In the present paper we investigate the local response of gauged thermal matter fields to a background electromagnetic field using the Euclidean (imaginary time) or Matsubara formalism (see, e.g., [13–15]). We formulate $(d+1)$-dimensional scalar electrodynamics on a hypercylinder of circumference $\beta = 1/T$ in the Euclidean time direction, choosing space to be flat and infinite. Our results reveal that a uniform background electric field $\vec{E}$ causes the thermal plasma to become non-uniform along the direction of $\vec{E}$, while the sea remains spatially uniform. This spacial non-uniformity has gone unnoticed in previous work [8–10] using global methods.

In [16] the same spatial non-uniformity of the thermal plasma was demonstrated for the case of the finite-temperature Schwinger model. Here we present analogous but more general calculations for thermal scalar fields, for which the discussion can easily be carried out in arbitrary spatial dimension. By coupling the scalar field to an arbitrary static gauge potential $A_\mu(\vec{x})$ we show that the characteristic effects arising from minimal coupling are common to all dimensions. Special attention is drawn to periodicity features related to gauge invariance and to the topology $S^1 \times R^d$ of Euclidean space–time. We then compute for the specific gauge potential $A_\mu = (Ex_1 + \text{const}, \vec{0})$, the thermal stress tensor $T^{(\beta)}_{\mu\nu}(x)$ and effective Lagrangian $\mathcal{L}_\beta(x)$. The dependence of the thermal plasma on the spatial coordinate x_1 is thereby made explicit.

A uniform background magnetic field $\vec{B}$ is also very briefly considered. $\vec{B}$ causes no spatial non-uniformity in the thermal plasma or sea unless a spatial direction perpendicular to $\vec{B}$ is compactified.

2. Thermal scalar field

Scalar electrodynamics is useful as a theoretical laboratory for studying gauge theory phenomena in arbitrary space–time dimension. We first compare the general problems of a thermal scalar field $\hat{\phi}$ coupled with an arbitrary static (i) non-gauge background potential $V(\vec{x})$ and (ii) gauge potential $A_\mu(\vec{x})$. We then specialize to the potential $A_\mu = (Ex_1 + \text{const}, \vec{0})$ for a uniform background electric field $\vec{E} = (E, 0, \ldots, 0)$ and compute explicitly the thermal stress tensor of $\hat{\phi}$. The case of a uniform magnetic field is also discussed briefly.

2.1. Scalar field in a static background Schrödinger potential

To set the stage we briefly review the case of a scalar quantum field interacting with a static background potential $V(\vec{x})$ in d-dimensional free space R^d. We wish to study the thermodynamical properties and vacuum Casimir energy of this system. To this end it will be convenient to work in the imaginary time or Matsubara formalism. Euclidean space–time is then a hyper-cylinder $S^1 \times R^d$. Correspondingly, we impose periodic boundary conditions in Euclidean time on the scalar field $\phi(x_0, \vec{x})$†,

$$\phi(x_0, \vec{x}) = \phi(x_0 + \beta, \vec{x}) \tag{2.1}$$

where $\beta = 1/T$ and T is the temperature. The spectral operator for the theory in question is $[-\partial_0^2 - \Delta + V(\vec{x})]$ with Δ the Laplacian in d dimensions. The vacuum and thermodynamical properties of the system can be computed from the bilocal heat kernel

$$h^{(\beta)}(t; x, y) = \sum_k \mathrm{e}^{-t\lambda_k^2} \phi_k(x) \phi_k^*(y)$$

† Throughout this section we use the Euclidean notation $x = \{x_\mu\} = (x_0, \vec{x})$.

where λ_k^2 and $\phi_k(x_0, \vec{x})$ are the eigenvalues and respective eigenfunctions of the spectral operator,

$$[-\partial_0^2 - \Delta + V(\vec{x})]\phi_k(x) = \lambda_k^2 \phi_k(x).$$

With $\phi(x)$ subject to the boundary condition (2.1) we have

$$\begin{aligned} \phi_k(x) &\to \phi_{mn}(x) = \frac{1}{\sqrt{\beta}} \mathrm{e}^{\mathrm{i}(\frac{2\pi m}{\beta})x_0} \varphi_n(\vec{x}) \\ \lambda_k^2 &\to \lambda_{mn}^2 = \left(\frac{2\pi m}{\beta}\right)^2 + \omega_n^2 \end{aligned} \tag{2.2}$$

where the spatial modes $\varphi_n(\vec{x})$ and associated spectrum $\{\omega_n^2\}$ are determined by the spatial mode equation

$$[-\Delta + V(\vec{x})]\varphi_n(\vec{x}) = \omega_n^2 \varphi_n(\vec{x}).$$

The Euclidean thermal Green function† then has the spectral representation

$$\begin{aligned} \langle \hat{\phi}(x)\hat{\phi}(y)\rangle_\beta &= \sum_{m,n} \frac{\phi_{mn}(x)\phi_{mn}(y)^*}{[(\frac{2\pi m}{\beta})^2 + \omega_n^2]} \\ &= \int_0^\infty \mathrm{d}t\, \frac{1}{\beta} \sum_m \mathrm{e}^{-t(\frac{2\pi m}{\beta})^2} \mathrm{e}^{\mathrm{i}\frac{2\pi m}{\beta}(x_0 - y_0)} \sum_n \mathrm{e}^{-t\omega_n^2} \varphi_n(\vec{x})\varphi_n^*(\vec{y}). \end{aligned}$$

We may perform the Matsubara sum by using the Jacobi identity [17]

$$\sum_{m=-\infty}^{\infty} \mathrm{e}^{-b(m-a)^2} = \sqrt{\frac{\pi}{b}} \sum_{l=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi^2 l^2}{b}} \mathrm{e}^{-\mathrm{i}2\pi a l} \tag{2.3}$$

with the result

$$\langle \hat{\phi}(x)\hat{\phi}(y)\rangle_\beta = \int_0^\infty \mathrm{d}t\, h(t; x, y)_{T=0} \sum_{l=-\infty}^{\infty} \mathrm{e}^{-\frac{l^2\beta^2}{4t}} \mathrm{e}^{\frac{l\beta}{2t}(x_0 - y_0)} \tag{2.4}$$

where

$$h(t; x, y)_{T=0} = \sqrt{\frac{1}{4\pi t}} \mathrm{e}^{-\frac{(x_0 - y_0)^2}{4t}} \sum_n \mathrm{e}^{-t\omega_n^2} \varphi_n(\vec{x})\varphi_n^*(\vec{y}) \tag{2.5}$$

is the $T = 0$ bilocal heat kernel of the operator $[-\partial_0^2 - \Delta + V(\vec{x})]$. Hence, for a static background potential the $T > 0$ Green function separates [18] into two distinct and well defined parts: the *virtual sea* part ($l = 0$ contribution) which is independent of T and coincides with the $T = 0$ Green function, and the *thermal gas* part ($l \neq 0$ contribution) which exhibits the full temperature dependence and vanishes exponentially as $T \to 0$:

$$\langle \hat{\phi}(x)\hat{\phi}(y)\rangle = \langle \hat{\phi}(x)\hat{\phi}(y)\rangle_{sea} + \langle \hat{\phi}(x)\hat{\phi}(y)\rangle_{gas}.$$

Now suppose that we have calculated from $\langle \hat{\phi}(x)\hat{\phi}(y)\rangle_{sea}$ the $T = 0$ vacuum stress tensor

$$T_{sea}^{\mu\nu} \equiv \langle \hat{T}^{\mu\nu}\rangle = \sum_n T_n^{\mu\nu} \tag{2.6}$$

as a (still to be renormalized) spatial mode sum. Repeating the calculation at finite temperature, from equation (2.4) we obtain

$$\begin{aligned} T^{(\beta)\mu\nu} &= \langle \hat{T}^{\mu\nu}\rangle_\beta \\ &= \sum T_n^{\mu\nu} \frac{1 + \mathrm{e}^{-\beta\omega_n}}{1 - \mathrm{e}^{-\beta\omega_n}} \\ &= T_{sea}^{\mu\nu} + \sum_n T_n^{\mu\nu} \frac{2}{\mathrm{e}^{\beta\omega_n} - 1} \\ &= T_{sea}^{\mu\nu} + T_{gas}^{\mu\nu}. \end{aligned} \tag{2.7}$$

† We denote operators by a 'hat'.

At finite temperature the mode sum is thus modified by the familiar Bose–Einstein distribution, in agreement with one's expectations. This completes our brief review of a usual Casimir problem at $T > 0$.

2.2. Scalar field in a static background gauge potential

Let us now consider a (massive or massless) scalar quantum field $\phi(x)$ coupled to a *static* Abelian background gauge potential $A_\mu(\vec{x})$. Again the quantum field $\phi(x_0, \vec{x})$ is required to satisfy the periodic boundary condition (2.1) in Euclidean time. The relevant spectral operator in this case is the gauged Laplacian in $d+1$ dimensions $-D_\mu^2$, where $D_\mu = \partial_\mu - \mathrm{i}A_\mu$ couples the quantum scalar field to a static Euclidean background gauge potential $A_\mu(\vec{x})$. (We have absorbed the electric charge into A_μ.) The thermodynamical and vacuum properties of the system can again be computed from the bilocal heat kernel

$$h^{(\beta)}(t; x, y) = \sum_k \mathrm{e}^{-t\lambda_k^2}\mathrm{e}^{-tM^2}\phi_k(x)\phi_k^*(y)$$

where λ_k^2 are the eigenvalues of $-D_\mu^2$, M is the mass and $\phi(x)$ is subject to the boundary condition (2.1). Periodicity in x_0 implies

$$\phi_k(x) \to \phi_{mn} = \frac{1}{\sqrt{\beta}}\mathrm{e}^{\mathrm{i}\frac{2\pi m}{\beta}x_0}\varphi_{mn}(\vec{x})$$

where $\varphi_{mn}(\vec{x})$ now satisfies the associated eigenvalue problem

$$[-(\vec{\nabla} - \mathrm{i}\vec{A}(\vec{x}))^2 + V_m(\vec{x})]\varphi_{mn}(\vec{x}) = \lambda_{mn}^2\varphi_{mn} \tag{2.8}$$

with

$$V_m(\vec{x}) = \left[A_0(\vec{x}) - \frac{2\pi m}{\beta}\right]^2. \tag{2.9}$$

Notice that the m-dependence of the Schrödinger-like background potential $V_m(\vec{x})$ leads to a coupling of spatial position $\vec{x}$ with the Matsubara frequencies. Equation (2.8) thus represent a different equation for each Matsubara frequency $\frac{2\pi m}{\beta}$, with n labelling the complete set of normalizable solutions $\{\varphi_{mn}\}$ of this Schrödinger problem for a given potential $V_m(\vec{x})$. The situation is thus very different from the scalar case discussed previously. It is characteristic of gauge theories and has very important consequences as we shall see.

On the cylinder $S^1 \times R^d$ we can always gauge a static $A_0(\vec{x})$ to the interval $[0, \frac{2\pi}{\beta}]$, but not in general to zero, if we respect the periodicity property (2.1). The only exception is when $A_0 = N\frac{2\pi}{\beta}$, in which case we may gauge A_0 to zero by performing the (allowed) gauge transformation $A_0 \to A_0 + \partial_0\lambda$ with $\lambda = (\frac{2\pi N}{\beta})x_0$.

Another way of stating this is to observe that in the exceptional case $A_0 = N\frac{2\pi}{\beta}$ the gauge transformation can be absorbed into the Matsubara index m via the transformation $m \to m+N$. For this reason $A_0(\vec{x})$ is always gauge equivalent to a configuration taking values in the range $[0, \frac{2\pi}{\beta}]$, so that effectively, at $T > 0$, the time component $A_0(\vec{x})$ of the Euclidean gauge potential becomes an angular variable. In the zero-temperature limit, on the other hand, we may always gauge $A_0(\vec{x})$ to zero. Indeed, the discrete Matsubara frequencies $k_0 = \frac{2\pi m}{\beta}$ become a continuous variable k_0 in the range $-\infty < k_0 < \infty$, and $A_0(\vec{x})$ may be absorbed into a shift in k_0 under the integral $\int \mathrm{d}k_0$.

Green function. Following the steps of the previous section, this time we have for the thermal Green function

$$\langle\hat{\phi}(x)\hat{\phi}(y)\rangle_\beta = \sum_{m,n}\frac{\phi_{mn}(x)\phi^*_{mn}(y)}{[\lambda^2_{mn}+M^2]}$$
$$= \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-tM^2}\frac{1}{\beta}\sum_m \mathrm{e}^{\mathrm{i}\frac{2\pi m}{\beta}(x_0-y_0)}h_m(t;\vec{x},\vec{y}) \tag{2.10}$$

where

$$h_m(t;\vec{x},\vec{y}) = \sum_n \mathrm{e}^{-t\lambda^2_{mn}}\varphi_{mn}(\vec{x})\varphi^*_{mn}(\vec{y}) \tag{2.11}$$

is the spatial bilocal heat kernel for the mth spatial background potential $V_m(\vec{x})$ in (2.9) . Notice that because of the coupling of Matsubara frequencies with the spatial modes the Green function can no longer be trivially separated into 'sea' and 'gas' contributions as in the potential problem discussed previously. However, this separation still exists, as the real-time approach to $T > 0$ QFT makes clear [9]. In the explicit examples that follow the separation into sea and gas components also eventually emerges in a natural fashion in the Matsubara approach.

Energy–momentum tensor. The symmetric canonical energy–momentum tensor for the complex scalar field $\hat{\phi}(x)$ coupled to a background A_μ in Euclidean space–time is formally given by

$$\hat{T}_{\mu\nu} = \tfrac{1}{2}[(D_\mu\hat{\phi})^\dagger(D_\nu\hat{\phi}) + (D_\nu\hat{\phi})^\dagger(D_\mu\hat{\phi})] - \delta_{\mu\nu}\hat{\mathcal{L}}$$

where

$$\hat{\mathcal{L}} = \tfrac{1}{2}[(D_\mu\hat{\phi})^\dagger(D^\mu\hat{\phi}) + M^2\hat{\phi}\hat{\phi}^\dagger].$$

Using the equation of motion $(D_\mu D_\mu - M^2)\hat{\phi} = 0$ we have for the divergence of $\hat{T}_{\mu\nu}$,

$$\partial_\mu\hat{T}_{\mu\nu} = F_{\mu\nu}\frac{\mathrm{i}}{2}[\hat{\phi}^\dagger(D_\mu\hat{\phi}) - (D_\mu\hat{\phi})^\dagger\hat{\phi}]$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and for the trace,

$$\hat{T}^\mu_\mu = \tfrac{1}{2}(1-d)(D_\mu\hat{\phi})^\dagger(D_\mu\hat{\phi}) - \tfrac{1}{2}(1+d)M^2\hat{\phi}^\dagger\hat{\phi}. \tag{2.12}$$

From (2.12) we see that $\hat{T}_{\mu\nu}$ is traceless only in the $d = 1$ spatial dimension for $M = 0$. Furthermore, we see that $\hat{T}_{\mu 0}$ is conserved if $F_{i0} = 0$, that is, if the electric field vanishes. This still allows for a static magnetic field, which makes sense, since a static magnetic field cannot do work on charges.

From (2.4) the thermal stress energy tensor $T^{(\beta)}_{\mu\nu} = \langle\hat{T}_{\mu\nu}\rangle_\beta$ is now easily written down. We have, for the separate components

$$\langle D_0\hat{\phi}(x)[D_0\hat{\phi}(x)]^\dagger\rangle_\beta = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-tM^2}\frac{1}{\beta}\left(\frac{2\pi}{\beta}\right)^2\sum_m[m-a(\vec{x})]^2 h_m(t;\vec{x},\vec{y}) \tag{2.13}$$

$$\langle D_i\hat{\phi}(x)[D_j\hat{\phi}(x)]^\dagger\rangle_\beta = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-tM^2}\frac{1}{\beta}\sum_m\lim_{\vec{x}\to\vec{y}}\{(D^x_i)^\dagger D^y_j h_m(t;\vec{x},\vec{y})\} \tag{2.14}$$

$$\langle D_0\hat{\phi}(x)[D_j\hat{\phi}(x)]^\dagger\rangle_\beta = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-tM^2}\frac{1}{\beta}\left(\frac{2\mathrm{i}\pi}{\beta}\right)\sum_m[m-a(\vec{x})]\lim_{\vec{x}\to\vec{y}}\{(D^y_j)^\dagger h_m(t;\vec{x},\vec{y})\} \tag{2.15}$$

$$\langle|\hat{\phi}(x)|^2\rangle_\beta = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-tM^2}\frac{1}{\beta}\sum_m h_m(t;\vec{x},\vec{x}) \tag{2.16}$$

where

$$a(\vec{x}) = \frac{\beta}{2\pi} A_0(\vec{x})$$

is the rescaled temporal component A_0.

We emphasize that A_0 is still Euclidean. Our ultimate goal is, of course, the Minkowski space–time tensor $T^{(\beta)}_{\mu\nu}$. Eventually we shall have to replace A_0 with $\mathrm{i}A_0$. We postpone doing this, however, until the very end of the calculation.

Of course, the above expressions giving the energy–momentum tensor in terms of the heat kernel need to be properly UV regularized. We see that these results differ essentially from the nongauge results given earlier, since in the present case $T_{\mu\nu}$ is expressed as a Matsubara sum over *nonidentical* spatial problems, the latter depending on Matsubara label m. In particular, an immediate separation into sea and thermal gas (or more properly, thermal plasma) contributions is, in general, not possible at this stage, as already mentioned.

2.2.1. Constant gauge potential. To begin with we consider a constant Euclidean background gauge potential

$$A_0 = \frac{2\pi}{\beta} a \qquad \vec{A} = 0. \tag{2.17}$$

Here a factor $\frac{1}{\beta}$ has been extracted to make a dimensionless. As we have already pointed out, on the cylinder $S^1 \times R^d$ this gauge potential cannot be gauged to zero; however, with the above parametrization, it is gauge equivalent to a potential with a in the range $0 \leqslant a \leqslant 1$. On the other hand, the spatial component $\vec{A}$ of a constant A_μ can always be gauged to zero on this cylinder, so that we may choose $\vec{A} = 0$.

Following our general notation we have in this case (for infinite volume the index n becomes the continuous momentum label $\vec{k}$)

$$\begin{aligned} \phi_{mk}(\vec{x}) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \\ \lambda^2_{mk} &= \left(\frac{2\pi}{\beta}\right)^2 (m-a)^2 + \vec{k}^2 \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} V_m(\vec{x}) &= \left(\frac{2\pi}{\beta}\right)^2 (m-a)^2 \\ h_m(t; \vec{x}, \vec{y}) &= \mathrm{e}^{-(\frac{2\pi}{\beta})^2 (m-a)^2} h_0(t; \vec{x} - \vec{y}) \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} h_0(t; \vec{x} - \vec{y}) &= \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \, \mathrm{e}^{-t\vec{k}^2} \mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})} \\ &= \frac{1}{(4\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{(\vec{x}-\vec{y})^2}{4t}} \end{aligned}$$

is the infinite volume, zero-temperature heat kernel of the free scalar field.

The factorization of the heat kernel h_m into an m-dependent and an m-independent factors now enables one to perform Matsubara sums explicitly by making use of the identity

$$\sum_{-\infty}^{\infty} (m-a)^2 \mathrm{e}^{-b(m-a)^2} = \frac{1}{2b} \sum_{-\infty}^{\infty} \mathrm{e}^{-b(m-a)^2} - 2\left(\frac{\pi}{b}\right)^{\frac{5}{2}} \sum_{n=1}^{\infty} n^2 \mathrm{e}^{-\frac{n^2\pi^2}{b}} \cos(2\pi a n) \tag{2.20}$$

obtained from (2.3) by differentiation with respect to b. One finds for the Euclidean T_{00}, after some simple algebra,

$$T_{00}^{(\beta)} = T_{00}^{sea} + T_{00}^{plasma} \tag{2.21}$$

where the temperature-independent part representing the sea is given by the UV-divergent integral

$$T_{00}^{sea} = \frac{1}{2}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-tM^2}(4\pi t)^{-(\frac{d+1}{2})}\left[\frac{d-1}{2t} - M^2\right] \tag{2.22}$$

and the temperature dependent part representing the thermal plasma carries all the dependence on the gauge potential and is finite:

$$T_{00}^{plasma} = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-tM^2}(4\pi t)^{-(\frac{d+1}{2})}\sum_{n=1}^{\infty}\cos(2\pi an)\left\{\frac{n^2\beta^2}{4t^2} + \frac{d-1}{2t} - M^2\right\}\mathrm{e}^{-\frac{n^2\beta^2}{4t}}. \tag{2.23}$$

The integral in (2.23) can be evaluated in terms of the modified Bessel function $K_\nu(z)$, but we shall not do so. It is important to observe that the thermal part vanishes exponentially as $T \to 0$, and that it exhibits the expected periodicity property in the Euclidean parameter a, in line with our earlier observation that a can always be chosen to lie in the interval [0, 1]. In [8–10] the authors presented $T > 0$ effective Lagrangians for thermal spinor fields coupled to a constant background A_0, displaying a similar cosine dependence on Euclidean A_0.

2.2.2. Constant electric field. Next we consider the linear Euclidean background potential

$$A_0(x_1) = Ex_1 + 2\pi a/\beta \qquad \vec{A} = 0 \tag{2.24}$$

corresponding to a constant background electric field $\vec{\mathcal{E}} = (-E, 0, \ldots, 0)$ in the x_1 direction. As will be seen, the limit $E \to 0$ is highly nontrivial. For that reason we have chosen to separately analyse the $E \neq 0$ problem here and the $E = 0$ problem in section 2.2.1 above.

Continuing with our general notation (where now $n \to (n, \vec{k}_\perp)$) we have the spatial modes

$$\varphi_{mn\vec{k}_\perp}(\vec{x}) = \varphi_n(x_m)(2\pi)^{-\frac{1}{2}(d-1)}\mathrm{e}^{\mathrm{i}\vec{k}_\perp\cdot\vec{x}_\perp} \tag{2.25}$$

with $\vec{k}_\perp = (k_2, \ldots, k_d)$ and $\vec{x}_\perp = (x_2, \ldots, x_d)$ representing momentum and position perpendicular to x_1. Inserting $\varphi_{mn\vec{k}_\perp}$ into the spatial mode equation (2.8) one obtains

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x_m^2} + E^2x_m^2\right]\varphi_n(x_m) = \epsilon_n\varphi_n(x_m) \tag{2.26}$$

where

$$x_m \equiv x_1 + \frac{2\pi}{\beta E}(a-m). \tag{2.27}$$

This is just the harmonic oscillator eigenvalue problem in Schrödinger theory with orthonormal eigenfunctions

$$\begin{aligned}\varphi_n(x_m) &= 2^{-n/2}\frac{1}{\sqrt{n!}}\left(\frac{E}{\pi}\right)^{\frac{1}{4}}\mathrm{e}^{-\frac{1}{2}Ex_m^2}H_n(\sqrt{E}x_m)\\ \epsilon_n &= 2E(n+\tfrac{1}{2}) \qquad n = 0, 1, 2, \ldots.\end{aligned} \tag{2.28}$$

Here $H_n(z)$ are Hermite polynomials satisfying $y'' - 2zy' + 2ny = 0$. In equation (2.8) the m-dependent backgrounds $V_m(x_1) = E^2x_m^2$ are identical harmonic oscillator potentials centered at equidistant positions $x_1 = (m-a)2\pi/\beta E$. As we shall see this periodic arrangement of identical potentials leads to a periodic structure along x_1 (with period $\Delta x_1 = 2\pi/\beta E$) in Euclidean $T_{\mu\nu}(x)$ and in other Euclidean local quantum functions. Therefore, the m

dependence of $\varphi_{mn\vec{k}_\perp}(\vec{x})$ resides entirely in the argument x_m of the harmonic oscillator wave function $\varphi_n(x_m)$, i.e. entirely in the position $x_1 = (m-a)2\pi/\beta E$ of the zero of $V_m(x_1)$. One consequence of this fact is that the spectrum of $-D^2$ given by

$$\lambda^2_{mn\vec{k}_\perp} = 2E(n+\tfrac{1}{2}) + \vec{k}^2_\perp \tag{2.29}$$

does *not* depend on the Matsubara label m. This is in sharp contrast with the nongauge scalar theory of section 2.1 with its spectrum (2.2), and with the constant A_μ problem above with spectrum (2.18).

The spatial heat kernel (2.11) constructed from the spatial modes (2.25) is

$$h_m(t;\vec{x},\vec{y}) = k_E(t;x_m,y_m)h_0(t;\vec{x}_\perp - \vec{y}_\perp) \tag{2.30}$$

where

$$\begin{aligned} k_E(t;x_m,y_m) &= \sum_{n=0}^{\infty} \mathrm{e}^{-t\lambda_n^2}\varphi_n(x_m)\varphi_n^*(y_m) \\ &= \left[\frac{E}{2\pi\sinh(2Et)}\right]^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}E(x_1-y_1)^2\coth(2Et)}\mathrm{e}^{-Ex_my_m\tanh(Et)} \end{aligned} \tag{2.31}$$

and h_0 is the free-space heat kernel for $d-1$ dimensions. The mode sum $\sum_n$ here has been performed with the help of the identity (see, e.g., [17] p 194)

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{2}\right)^n H_n(x)H_n(y) = (1-z^2)^{-1/2}\exp\left\{\frac{z}{1-z^2}[2xy(1-z) - z(x-y)^2]\right\}.$$

Alternatively, one can use known formulae for the propagator in the harmonic oscillator problem. Note that in the limit $E\to 0$ the heat kernel (2.31) smoothly becomes the one-dimensional free-space heat kernel as it should

$$k_E(t;x_m,y_m) \to \frac{1}{\sqrt{4\pi}}\mathrm{e}^{-(x_1-y_1)^2/4t} \qquad E\to 0.$$

The limit $E\to 0$ is nonetheless far from being uniform: as $E\to 0$ the potentials $V_m(x_1) = E^2x_m^2 \to (2\pi m/\beta)^2$ change into m-dependent constants (like mass terms) independent of x_1. The background returns to the constant potential $A_\mu = (2\pi a/\beta, \vec{0})$ of the preceeding section with its spectrum $\lambda^2_{mk} = (m-a)^2(2\pi/\beta)^2 + \vec{k}^2$. Remarkably, the dependence on m so conspicuously absent from the $E>0$ spectrum $\lambda^2_{mnk_\perp}$ re-enters the $E=0$ spectrum.

The Green function (2.10) and local quantities derived from it possess the sea + plasma structure one expects to find. Let us display this for the Green function (2.10) in the limit $y\to x$;

$$\langle|\hat{\phi}(x)|^2\rangle_\beta = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-tM^2}(4\pi t)^{-d/2}\left[\frac{2Et}{\sinh(2Et)}\right]^{\frac{1}{2}}\frac{1}{\beta}\sum_m \mathrm{e}^{-Ex_m^2\tanh(Et)} \tag{2.32}$$

where, using the identity (2.3), the Matsubara sum can be evaluated with the result

$$\frac{1}{\beta}\sum_{m=-\infty}^{\infty}\mathrm{e}^{-Ex_m^2\tanh Et} = \left[\frac{E}{4\pi\tanh Et}\right]^{\frac{1}{2}}\sum_{n=-\infty}^{\infty}\mathrm{e}^{-n^2\beta^2E/4\tanh Et}\mathrm{e}^{-\mathrm{i}n(2\pi a + x_1\beta E)}. \tag{2.33}$$

Thus, we find

$$\langle|\hat{\phi}(x)|^2\rangle_{sea} = \langle|\hat{\phi}(x)|^2\rangle_{T=0} = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-tM^2}(4\pi t)^{-\frac{d-1}{2}}\frac{E}{4\pi\sinh Et} \tag{2.34}$$

$$\langle|\hat{\phi}(x)|^2\rangle_\beta^{plasma} = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-tM^2}(4\pi t)^{-\frac{d-1}{2}}\frac{E}{4\pi\sinh Et}$$

$$\times \sum_{n=1}^{\infty} 2\cos n(2\pi a + x_1 \beta E)\mathrm{e}^{-n^2\beta^2 E/4\tanh Et}. \tag{2.35}$$

One easily verifies that $\langle|\hat{\phi}(x)|^2\rangle_{sea}$ coincides with the corresponding $T = 0$ quantity $\langle|\hat{\phi}(x)|^2\rangle_{T=0}$ as it should. Of course, this function needs UV renormalization. All dependence on temperature is in $\langle|\hat{\phi}(x)|^2\rangle_{\beta}^{plasma}$. The latter function is finite and it vanishes exponentially as $T \to 0$. Moreover, it is periodic in x_1 with period $\Delta x_1 = 2\pi/\beta E$, reflecting the equidistant arrangement of potentials $V_m(x_1) = E^2 x_m^2$.

We now proceed to the straightforward calculation of $T_{\mu\nu}^{(\beta)}$. Using equations (2.13)–(2.16) one easily verifies

$$T_{00}^{(\beta)}(x) = T_{00}^{sea} + T_{00}^{plasma}(x_1) \tag{2.36}$$

where

$$T_{00}^{sea} = \frac{1}{2}\int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-tM^2}(4\pi t)^{-\frac{1}{2}(d-1)} \frac{E}{4\pi \sinh Et}\left[\frac{d-1}{2t} - M^2\right] \tag{2.37}$$

and

$$T_{00}^{plasma}(x_1) = \int_0^{\infty} \mathrm{d}t\mathrm{e}^{-tM^2}(4\pi t)^{-\frac{1}{2}(d-1)} \frac{E}{4\pi \sinh Et} \sum_{n=1}^{\infty} \mathrm{e}^{-n^2\beta^2 E/4\tanh Et} \cos n(2\pi a + x_1\beta E)$$
$$\times \left[\frac{d-1}{2t} - M^2 + n^2\left(\frac{E\beta}{2\sinh Et}\right)^2\right]. \tag{2.38}$$

The above results for the virtual sea and thermal plasma have been obtained by performing the Matsubara sums in equation (2.36) with the help of the identities (2.3), (2.20). For the reader's convenience we give the form in which the latter identity is used here:

$$\frac{E^2}{\beta}\sum_{m=-\infty}^{\infty} x_m^2 \mathrm{e}^{-x_m^2 E\tanh Et} = \left[\frac{E}{4\pi \tanh Et}\right]^{\frac{1}{2}} \frac{E}{2\tanh Et}$$
$$\times\left\{\sum_{n=-\infty}^{\infty} \mathrm{e}^{-n^2\beta^2 E/4\tanh Et}\mathrm{e}^{-\mathrm{i}n(2\pi a + x_1\beta E)}\right.$$
$$\left. - \frac{\beta^2 E}{\tanh Et}\sum_{n=1}^{\infty} n^2 \mathrm{e}^{-n^2\beta^2 E/4\tanh Et}\cos n(\beta E x_1 + 2\pi a)\right\}. \tag{2.39}$$

Of course, T_{00}^{sea} needs UV renormalization. In the limit $E \to 0$, T_{00}^{sea} and T_{00}^{plasma} above smoothly become the $E = 0$ functions (2.22), (2.23).

As expected, T_{00}^{sea}, although a function of E, is independent of position: the uniform electric field leaves the virtual sea spatially uniform. Physically, this seems reasonable. Virtual particles do not have the prolonged existence needed to participate in e.g. thermal equilibrium. This is why the sea remains temperature independent.

Things are different for the thermal plasma. The particles of the thermal plasma do have prolonged existence, and they do participate in thermal equilibrium. Moreover, these charged particles feel the background gauge potential itself.

2.2.3. Continuation to Minkowski space–time. Continuing our results to Minkowski space–time by setting $E = \mathrm{i}\mathcal{E}$, $a = \mathrm{i}\frac{\mu\beta}{2\pi}$, where $\mathcal{E}$ represents the Minkowski electric field and μ is a chemical potential for the scalar field (see, e.g., [19]), we finally obtain

$$T_{00}^{sea} = \frac{1}{2}\int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-tM^2}(4\pi t)^{-\frac{1}{2}(d-1)} \frac{\mathcal{E}}{4\pi \sin \mathcal{E}t}\left[\frac{d-1}{2t} + M^2\right] \tag{2.40}$$

and

$$T_{00}^{plasma}(x_1) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-tM^2}(4\pi t)^{-\frac{1}{2}(d-1)} \frac{\mathcal{E}}{4\pi \sin \mathcal{E}t} \sum_{n=1}^{\infty} \mathrm{e}^{-n^2\beta^2 \frac{\mathcal{E}}{4\tan \mathcal{E}t}} \cosh n\beta(\mu + x_1\mathcal{E})$$
$$\times \left[\frac{d-1}{2t} + M^2 + n^2 \left(\frac{\mathcal{E}\beta}{2\sin \mathcal{E}t} \right)^2 \right]. \tag{2.41}$$

The integral (2.40) for the energy density of the virtual sea is singular at $t = 0$ (the usual free space UV divergence) and at the points $t = q(\frac{\pi}{\mathcal{E}}), q = 1, 2, 3 \ldots$. Schwinger, in his classic treatment [2] of the $T = 0$ version of this problem, interprets such additional singularities in t in terms of pair production from the sea by the electric field. We have nothing to add to this; our concern is the effect of a constant electric field on the thermal plasma. The integral (2.41) for the energy density of the thermal plasma is *non-singular*. Indeed, in other Casimir-like contexts [18], the functions representing the thermal plasma are always finite and require no renormalization. This continues to be true for static background electromagnetic fields interacting with charged scalar fields.

For comparison with Schwinger's calculation [2] we sketch our derivation of his result for the effective Lagrangian of the $T = 0$ scalar field. Define the space–time zeta function

$$Z(s|x,x)_\beta \equiv \mu^{2s} \sum_{m,n} [\lambda^2 + M^2]^{-s} |\phi_{mn}(x)|^2$$
$$= \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \mathrm{e}^{-tM^2} \frac{1}{\beta} \sum_m h_m(t|\vec{x}, \vec{x})$$

where h_m is the heat kernel (2.11), in general. For the specific heat kernel (2.30), (2.31), the Matsubara sum can again be performed using equation (2.33). Defining the thermal Lagrangian $\mathcal{L}_\beta = \mathcal{L}_{sea} + \mathcal{L}^\beta_{plasma}$ by $\mathcal{L}_\beta = -Z'(0|x,x)_\beta$, where the prime denotes differentiation with respect to s, one finds

$$\mathcal{L}_{sea} = -\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-tM^2}(4\pi t)^{-\frac{1}{2}(d+1)} \frac{\mathcal{E}}{\sin \mathcal{E}t} \tag{2.42}$$

$$\mathcal{L}^\beta_{plasma}(x_1) = -\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-tM^2}(4\pi t)^{-\frac{1}{2}(d+1)} \frac{\mathcal{E}}{\sin \mathcal{E}t} \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{n^2\beta^2\mathcal{E}}{4\tanh \mathcal{E}t}} 2\cosh n\beta(\mu + \mathcal{E}x_1). \tag{2.43}$$

The plasma contribution $\mathcal{L}_{plasma}$ vanishes exponentially as the temperature goes to zero. The temperature independent part $\mathcal{L}_{sea}$ agrees for $d = 3$ with Schwinger's result for a $T = 0$ scalar field in a constant background electric field [2].

The integral (2.42) diverges at $t = 0$ and needs UV regularization. It also exhibits singularities at $t = q(\frac{\pi}{\mathcal{E}})$, $q = 1, 2, 3 \ldots$. As shown by Schwinger [2], removal of the latter singularities generates an imaginary part in the renormalized $\mathcal{L}_{sea}$, which corresponds to particle production from the virtual sea by the electric field. In contrast, the integral (2.43), representing the plasma contribution to $\mathcal{L}_\beta$ is finite and real. Thus, finite temperature has no effect on the rate of particle production by the electric field [8, 10]. We believe this should be the case, since particle production occurs from the virtual sea, which is independent of T.

A final comment concerns $\cosh n\beta A_0(\vec{x})$ under the sum in equation (2.41) and in other local quantum functions characterizing the thermal plasma. Clearly, what is displayed is the Minkowski gauge potential A_0 behaving as a chemical potential. It costs potential energy $\epsilon = \pm A_0(\vec{x})$ to create a scalar particle/antiparticle at location $\vec{x}$. The interpretation of a constant term $\mathrm{i}\mu$ in the Euclidean gauge potential A_0 (or μ in the Minkowski gauge potential A_0) as a chemical potential was noticed long ago (see, e.g., [19] and early references therein). Formulae such as equation (2.41) extend this idea to arbitrary functions $A_0(\vec{x})$. But notice that other factors under the sum in equation (2.41) also depend on the electric field. Thus, equation (2.41) does not result simply from standard statistical mechanics considerations.

Previous studies of thermal fields coupled to a uniform electric field [8–10], using real-time global methods, could not pick up the $\cosh n\beta(\mu + \mathcal{E}x_1)$ dependence in $\mathcal{L}^{\beta}_{plasma}(x_1)$.

The exponential dependence on the direction parallel to the electric field reflects the instability of the physical system under consideration, and suggests the following physical picture: pair production from the sea is independent of the temperature and uniform throughout space, since pairs are produced essentially at a point and therefore electrostatic energy is not involved. The thermal plasma, on the other hand, is sensitive to the external voltage $A_0 = \mu + \mathcal{E}x_1$, which is seen from (2.43) to play the role of a position-dependent chemical potential. This is the situation in infinite space, as discussed here. A possible physical interpretation of the system in question is that the time interval has not been sufficiently long for the applied electric field to be depleted by particle production [10]. If, on the other hand, the applied electric field were that of two isolated charged plates, the electric field would be depleted after some time, eventually leading to a stable condition of the plasma. As mentioned in [7], this final situation could be formally described by the introduction of a space-dependent chemical potential.

It would be interesting to see how our x_1-dependent result is reproduced by real-time methods. In particular, the relation between our local results and the global results of [8–10], are by no means clear and deserve further study. This is, however, outside the scope of the present paper.

2.2.4. Constant magnetic field. To investigate the effect of a uniform background magnetic field on the virtual sea and thermal gas of a scalar field it is of particular interest to consider the case $d = 3$ spatial dimensions. We choose for the static background potential $A_\mu = (0, 0, Bx_1, 0)$ leading to the magnetic field $\vec{B} = (0, 0, B)$. Following our general notation we then have for the spatial modes (now $n \to (n, k_2, k_3)$)

$$\phi_{mnk_2k_3}(\vec{x}) = \frac{1}{2\pi}\mathrm{e}^{\mathrm{i}(k_2x_2+k_3x_3)}\varphi_n(x_{k_2})$$

where k_2, k_3 are continuous momentum labels in finite space and

$$x_{k_2} = x_1 + \frac{k_2}{B}.$$

The modes $\varphi_n(x_{k_2})$ are the harmonic oscillator wavefunctions (2.28) satisfying

$$[-\partial_1^2 + B^2x_{k_2}^2]\phi_n(x_{k_2}) = 2B(n + \tfrac{1}{2})\phi_n(x_{k_2}).$$

The eigenvalues of $-D^2$ are now

$$\lambda^2_{mnk_2k_3} = \left(m\frac{2\pi}{\beta}\right)^2 + 2B\left(n + \frac{1}{2}\right) + k_3^2$$

and are independent of k_2.

It is already apparent that the uniform magnetic field does not introduce spatial non-uniformity into either the virtual sea or the thermal gas. Indeed, all mode sums involve an integration in k_2 over the infinite interval $[-\infty, \infty]$. Since k_2 is a continuous variable we can perform the shift $k_2 \to k_2 - Bx_1$ in the integration variable, thereby absorbing the x_1 dependence into the integration. Thus $T_{\mu\nu}$ and other local quantum functions will not depend on x_1. See [6] for considerably more detail, especially on $\mathcal{L}_\beta$.

If, however, we compactify the x_2 direction perpendicular to the magnetic field to a circle of perimeter L, we are led to a real-time problem paralleling the Euclidean one with constant electric field discussed above. Compact x_2 corresponds to discrete momenta $k_2 = p(\frac{2\pi}{L})$, with p running over all integers (as in the case of the Matsubara index). The harmonic oscillator mode equation above becomes

$$[-\partial_1^2 + B^2 x_p^2]\varphi_n(x_p) = 2B(n + \tfrac{1}{2})\varphi_n(x_p)$$

where $x_p = x_1 + p\frac{2\pi}{BL}$. Again we have an infinite set of harmonic oscillator potentials equally spaced at intervals $\Delta x_1 = \frac{2\pi}{BL}$ along the x_1-axis. Consequently, local quantities such as $T_{\mu\nu}$ are periodic in x_1 with period Δx_1. Clearly one is encountering something akin to the quantum Hall effect.

3. Conclusion

By calculating the thermal stress tensor $T^{(\beta)}_{\mu\nu}(x)$ of a $T > 0$ scalar quantum field in the presence of a constant electric field, we found that the thermal plasma distribution of the scalar field becomes non-uniform along the direction of $\vec{E}$, while the virtual sea remains uniform. On the other hand, a background uniform magnetic field does not lead to a spatial non-uniformity in either plasma or sea unless we compactify one of the spatial directions perpendicular to the magnetic field (say x_2) to a circle. In this case periodicity of the energy density distribution is found along a direction perpendicular to both x_2 and to the magnetic field. In both cases (electric and magnetic field) this spatial non-uniformity can be traced to the underlying gauge invariance of the theory and the possibility of mapping $A_0(\vec{x})$ and $A_2(\vec{x})$ into the intervals $[0, \frac{2\pi}{\beta}]$ and $[0, \frac{2\pi}{L}]$, respectively, by a bona fide gauge transformation.

Our exact results for a constant background electromagnetic field have a natural extension to arbitrary static background fields $\vec{E}(\vec{x})$ and $\vec{B}(\vec{x})$. One relevant equation to consult is equation (2.8). There we see that the space components of the vector potential are decoupled from the Matsubara index m. For an arbitrary static magnetic field $\vec{B} = \nabla \times \vec{A}$ with $A_0 = 0$ (recall that at finite temperature $A_0 = 0$ cannot be generally achieved by a gauge transformation) one simply makes the replacements $\lambda^2_{mn} \to (\frac{2\pi m}{\beta})^2 + \omega_n^2$ and $\varphi_{mn} \to \varphi_n$ in equation (2.8). The spatial mode equation then becomes

$$[-(\nabla - \mathrm{i}\vec{A})^2]\varphi_n(\vec{x}) = \omega_n^2\varphi_n(\vec{x}).$$

For arbitrary $\vec{B}(\vec{x})$ one thus has a situation much like the non-gauge theory of section 2.1, leading to some non-uniform distribution of both the sea and plasma components comparable with what physical boundaries would cause.

The situation is quite different when A_0 is non-zero. Then the static background field affects the sea and plasma quite differently. While our calculations apply strictly only for constant $\vec{E}$, one would expect for any electric field which is only weakly dependent on $\vec{x}$ a roughly similar response from the plasma, and a nearly uniform distribution of the sea component. Mathematically, we have an infinite set of eigenvalue equations with a *different* Schrödinger-like background potential $V_m(\vec{x}) = [A_0(\vec{x}) - 2\pi m/\beta]^2$ for each Matsubara frequency. Note that $m \to m + N$ corresponds to performing an allowed gauge transformation with gauge function $\lambda = x_0(2\pi N/\beta)$, and that $V_{m+N}(\vec{x})$ and $V_m(\vec{x})$ are connected by this gauge transformation. This situation differs fundamentally from the non-gauge case, where the potential $V(\vec{x})$ does not bear the label m. Since Green functions, the energy–momentum tensor, etc, are given in terms of equally weighted sums over all the individual problems labelled by m, they are explicitly gauge invariant.

In general the diagonal heat kernel of a scalar or fermion quantum field at finite temperature $T > 0$ in a *static* background should factorize in the following way:

$$h^{(\beta)}(t; x, x) = h(t; x, x)_{T=0}[1 + f(t; x; T)]$$

where $h(t; x, x)_{T=0}$ is the zero-temperature heat kernel for the static background, and $f(t, x, T)$ is some function of the temperature T, the diffusion or 'proper' time t and the spatial position $\vec{x}$. This function f vanishes exponentially as either $T \to 0$ or $t \to 0$. The factorization above is expected because $h^{(\beta)}(t; x, x)$ separates quite generally for a static background into an UV divergent sea part, and an UV finite gas part [9, 18]

$$h^{(\beta)}(t; x, x) = h(t; x, x)_{sea} + h(t; \vec{x})_{gas}$$

where we make the identification

$$h(t; x, x)_{sea} = h(t; x, x)_{T=0}.$$

Defining $f(t; x; T)$ by

$$h(t; \vec{x})_{gas} = f(t; x; T)h(t; x, x)_{T=0}$$

leads to the factorization above. Known properties of $h(t; \vec{x})_{gas}$ then imply the stated properties of f.

It is a matter of some interest to study the function $f(t, \vec{x}; T)$. Let us list the explicit examples computed in this paper.

- Non-gauge scalar theory:

$$f = 2\sum_{n=1}^{\infty} \mathrm{e}^{-n^2\beta^2/4t}.$$

- Gauged scalar theory with $A_0 = \mathcal{E}x_1 + \text{const}$:

$$f = 2\sum_{n=1}^{\infty} \mathrm{e}^{-n^2\beta^2\mathcal{E}/4\tan\mathcal{E}t} \cosh n\beta A_0.$$

From the gauge theory example we see that $f(t, \vec{x}; T)$ in general depends on A_0 when a gauge potential background is involved. Investigation of this dependence for arbitrary $A_0(x)$ is an interesting mathematical problem. Work on this is in progress.

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